

Nonextensive Statistics and Multiplicity Distribution in Hadronic Collisions

C.E. Aguiar and T. Kodama

Instituto de Física, Universidade Federal do Rio de Janeiro
Cx.P. 68528, Rio de Janeiro, 21945-970, RJ, Brasil

Abstract

The multiplicity distribution of particles in relativistic gases is studied in terms of Tsallis' nonextensive statistics. For an entropic index $q > 1$ the multiplicity distribution is wider than the Poisson distribution with the same average number of particles, being similar to the negative binomial distribution commonly used in phenomenological analysis of hadron production in high-energy collisions.

1 Introduction

It is always amusing to recognize that the concepts of equilibrium statistical physics, and hence of thermodynamics, can be applied in (at first sight) unexpected branches of physics. For example, it has long been known that several features of multiparticle production in high-energy hadronic collisions are well described by thermostatistical models [1, 2, 3]. More recently, it has been shown that a simple thermal description of particle abundances works quite well for the hadrons produced in elementary electron-positron collisions [4, 5]. This indicates that the hadronic system reaches statistical equilibrium, at least for some observables. One might try to justify this in terms of the large number of degrees of freedom in the final state.

Thermostatistics is also an important issue in relativistic heavy ion physics. The main focus of recent investigations in this field is the production and identification of the quark-gluon plasma, a phase of nuclear matter consisting of unconfined quarks and gluons. Thermostatistical aspects play a central role in these investigations, as it is expected that some form of statistical equilibrium is attained in high-energy collisions of two heavy nuclei, at least in the central region [6, 7].

One of the signatures of "thermal" multiparticle production is the exponential form of the transverse energy distribution of the produced hadrons. We are tempted to interpret the slope parameter of this distribution in terms of a temperature of the final state. In fact, up to the ISR energy region, the transverse

energy distribution is consistent with such an interpretation. The same happens to the observed hadron abundances.

However, at higher energies, the thermal interpretation of the transverse energy spectra should be modified. The SPS experiments revealed a significant deviation from the exponential transverse energy distribution, together with the violation of several scaling laws. Usually, these aspects are understood in terms of the onset of hard QCD processes such as the formation of mini-jets. Furthermore, even if local thermal equilibrium is attained, dynamical effects such as collective flow may entangle with the thermal transverse spectra. Consequently, for these energies, a pure statistical description to the overall system cannot be applied.

Recently, it was pointed out [8, 9, 10, 11] that the nonexponential behavior of the transverse energy distribution in high-energy multiparticle production processes (including electron-positron annihilation) can be described rather well by the nonextensive thermostatistics of Tsallis [12]. In addition to the excellent quality of the fit to transverse energy spectra, an interesting point of the extended statistical approach to multiparticle production is that the “temperature” stays almost constant, independent of the incident energy, conveniently recovering the classical Hagedorn scenario [2, 3, 8]. This opens the possibility of reviving the statistical description of the complex dynamics of hadronic collisions in a unified way, including the onset of semi-hard QCD processes, provided that we generalize the concept of statistical equilibrium. Differently from the standard statistical mechanics of Boltzmann and Gibbs, the Tsallis nonextensive thermostatistics contains one extra parameter, the “entropic index” q , originated in the definition of the entropy. The physical origin of this parameter is not yet completely understood and a lot of discussions are in course. It can be attributed to various factors, like the dynamical suppression of certain domain of the phase space (multifractal structure near the critical point), the presence of long range forces, or fluctuations in a small system [13, 14]. For an entropic index $q = 1$ Tsallis statistics reduces to the Boltzmann-Gibbs theory. If $q > 1$, rare events are enhanced relative to the Boltzmann-Gibbs case. If $q < 1$, frequent events are privileged. So far, all phenomenological applications of Tsallis statistics to high-energy collisions have found $q > 1$ in these processes. Some discussion on the possible origin of nonextensive behavior in hadronic systems can be found in Refs. [15, 16].

The transverse energy distribution is not the only probe of the formation of a locally thermalized source of particles in high-energy collisions. Another important observable is the multiplicity distribution of the produced hadrons. The negative binomial distribution [17]

$$P_N = \frac{\Gamma(N+k)}{\Gamma(N+1)\Gamma(k)} \frac{(\overline{N}/k)^N}{(1+\overline{N}/k)^{N+k}}, \quad (1)$$

is often used to express hadron abundances in high-energy processes, where N is the number of particles and \overline{N} the average multiplicity. The k parameter is

related to the variance $D^2 = \overline{N^2} - \overline{N}^2$ of the distribution by

$$D^2 = \overline{N} + \frac{\overline{N}^2}{k} . \quad (2)$$

Although the negative binomial distribution is known to arise from some specific processes, such as Bose-Einstein particles with different sources [18], its use in high-energy multiparticle production is rather phenomenological.

In this paper, we discuss how the generalized statistical mechanics of Tsallis (with $q > 1$) affects the multiplicity distribution in hadronic collisions. We show that for q close to unity the distribution of particle numbers in a high-energy gas is approximately negative binomial. The Boltzmann-Gibbs and Tsallis statistics are applied to a simple meson gas model, and the results are compared to experimental data.

2 Nonextensive q statistics

First, let us review briefly Tsallis' statistics and derive some formulas necessary for our discussion. Tsallis generalized the usual Boltzmann-Gibbs thermostatics by introducing the “ q -entropy”

$$S = \frac{1 - \sum_a p_a^q}{q - 1} , \quad (3)$$

where p_a is the probability associated with microstate a , normalized as

$$\sum_a p_a = 1 . \quad (4)$$

Tsallis also introduced “ q -biased” averages of observables,

$$\langle O \rangle = \frac{1}{\mathcal{C}_q} \sum_a O_a p_a^q , \quad (5)$$

the normalization factor being

$$\mathcal{C}_q = \sum_a p_a^q . \quad (6)$$

The q -biased microstate probability

$$\tilde{p}_a = \frac{p_a^q}{\mathcal{C}_q} \quad (7)$$

is the probability to be used in the calculation of physical quantities. The entropic index q is a real number, and in the limit $q \rightarrow 1$ the Boltzmann-Gibbs-Shannon entropy is recovered. In this limit the q -average also reduces to the usual one. The equilibrium probabilities p_a are determined by maximizing the

entropy under appropriate constraints. These are the normalization condition (4), and the fixed (average) value of the energy E and of any other relevant conserved quantity. We assume for simplicity that there is only one such quantity, a conserved charge Q . The variational principle is then

$$\delta S + \alpha \sum_a \delta p_a - \beta_T \delta E + \gamma \delta Q = 0 , \quad (8)$$

where

$$E = \sum_a E_a \tilde{p}_a , \quad (9)$$

$$Q = \sum_a Q_a \tilde{p}_a , \quad (10)$$

and E_a and Q_a are the energy and charge of the microstate a . The constants α , β_T and γ are Lagrange multipliers. The later two are associated with the temperature T and chemical potential μ (in the sense of the second law of thermodynamics) as

$$\beta_T = \frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)_{Q,V} , \quad (11)$$

$$\gamma = \frac{\mu}{T} = - \left(\frac{\partial S}{\partial Q} \right)_{E,V} , \quad (12)$$

where V is the volume occupied by the system. Solving the variational equation (8) we obtain the Tsallis distribution,

$$\tilde{p}_a = \frac{1}{Z_q} \{ \exp_q [-\beta (E_a - \mu Q_a)] \}^q , \quad (13)$$

where we have defined the “ q -exponential function”, \exp_q , by

$$\exp_q(A) \equiv [1 - (q-1)A]^{-1/(q-1)} . \quad (14)$$

We have also introduced the generalized partition function Z_q as

$$Z_q(\beta, \mu, V) = \sum_a \{ \exp_q [-\beta (E_a - \mu Q_a)] \}^q . \quad (15)$$

Note that in the limit of $q \rightarrow 1$ we have

$$\lim_{q \rightarrow 1} \exp_q(A) = e^A \quad (16)$$

so that Eqs. (13) and (15) reduce to the corresponding quantities of the usual Boltzmann-Gibbs statistical mechanics. The parameter β appearing in these equations is not the inverse temperature Lagrange multiplier $\beta_T = 1/T$; β is related to temperature T by

$$T = \frac{\beta^{-1} + (q-1)(E - \mu Q)}{1 + (1-q)S} . \quad (17)$$

The Tsallis non-intensive temperature T meets some difficulties when confronted with the zeroth law of thermodynamics. This point has been investigated recently [19], and it was shown that the quantity

$$\tilde{T} = \beta^{-1} + (q-1)(E - \mu Q) , \quad (18)$$

sometimes called the “physical” temperature, can provide a better characterization of thermal equilibrium.

Instead of using the chemical potential μ to control the average charge Q , as in the grand canonical approach outlined above, it is sometimes necessary to impose charge conservation exactly. This is particularly important for small values of Q , when fluctuations around the average become significant. Then a canonical treatment is preferable, in which case the generalized partition function for a fixed charge Q is given by

$$Z_q(\beta, Q, V) = \sum_a \delta(Q - Q_a) [\exp_q(-\beta E_a)]^q \quad (19)$$

where $\delta(Q - Q_a)$ is a Kronecker delta. The generalized canonical probability is

$$\tilde{p}_a = \frac{\delta(Q - Q_a)}{Z_q(\beta, Q, V)} [\exp_q(-\beta E_a)]^q . \quad (20)$$

3 Integral representation of the Tsallis distribution

For further development of the theory, it is useful to introduce the following integral representation, valid for $q > 1$,

$$[\exp_q(A)]^q = \int_0^\infty dx G(x) e^{xA} , \quad (21)$$

where

$$G(x) = \frac{(\nu x)^\nu}{\Gamma(\nu)} e^{-\nu x} \quad (22)$$

with $\nu = 1/(q-1)$. Since $G(x) \geq 0$ and

$$\int_0^\infty G(x) dx = 1, \quad (23)$$

the function $G(x)$ can be considered as the probability distribution of the variable $x \in [0, \infty)$. The maximum of $G(x)$ is at $x = 1$, and the moments of x are

$$\overline{x^n} = \int_0^\infty x^n G(x) dx = \frac{(\nu+n)(\nu+n-1)\cdots(\nu+1)}{\nu^n} . \quad (24)$$

In particular, we have

$$\overline{x} = \frac{\nu+1}{\nu} = q , \quad (25)$$

$$\overline{x^2} = \frac{(\nu+1)(\nu+2)}{\nu^2} = q(2q-1) , \quad (26)$$

so that the width of $G(x)$ is

$$\sigma_x = \sqrt{x^2 - \overline{x^2}} = \sqrt{q(q-1)} . \quad (27)$$

For $q \rightarrow 1$, $G(x)$ tends to the Dirac delta function $\delta(x-1)$.

Using the integral representation (21), we can express the generalized partition function (15) as

$$Z_q(\beta, \mu, V) = \int_0^\infty dx G(x) Z(x\beta, \mu, V) , \quad (28)$$

where

$$Z(\beta, \mu, V) = \sum_a \exp[-\beta(E_a - \mu Q_a)] \quad (29)$$

is the Boltzmann-Gibbs partition function.

The integral representation of the q -exponential function is useful for developing approximations when q is close to unity. As we have seen, in this case the function $G(x)$ is sharply peaked at $x=1$. Thus, in any expression of the form

$$I = \int_0^\infty dx G(x) e^{f(x)} , \quad (30)$$

we may expand the exponent $f(x)$ around $x=1$ as

$$f(x) \simeq f(1) + f'(1)(x-1) + \dots , \quad (31)$$

and obtain

$$\begin{aligned} I &\simeq e^{f(1)-f'(1)} \int_0^\infty dx G(x) e^{xf'(1)} \\ &= e^{f(1)-f'(1)} \{\exp_q[f'(1)]\}^q \\ &\simeq \exp\{f(1) + (q-1)f'(1)[1+f'(1)/2]\} . \end{aligned} \quad (32)$$

In the last step we have used

$$[\exp_q(A)]^q \simeq \exp[A + (q-1)(A + A^2/2)] , \quad (33)$$

valid for $q \simeq 1$. Applying these approximations to Eq. (28), the generalized partition function $Z_q(\beta, \mu, V)$ can be written in terms of the Boltzmann-Gibbs function $Z(\beta, \mu, V)$ as

$$Z_q(\beta, \mu, V) \simeq Z(\beta, \mu, V) \exp\{(q-1)[- \beta(E - \mu Q) + \beta^2(E - \mu Q)^2/2]\} , \quad (34)$$

where

$$E - \mu Q = -\frac{\partial \ln Z}{\partial \beta} . \quad (35)$$

Integral representations of \exp_q also exist for $q < 1$ [20, 21]. We will not explore these here because, as mentioned in the Introduction, $q > 1$ seems to be the case of interest in high-energy collisions.

4 Particle multiplicity distribution

The probability that the system has exactly N particles is given, in the Tsallis statistics, by

$$P_N = \sum_a \delta(N - N_a) \tilde{p}_a \quad (36)$$

where N_a is the particle number in state a . Defining the partition function restricted to N particles,

$$Z_q^{(N)}(\beta, \mu, V) = \sum_a \delta(N - N_a) \{ \exp_q [-\beta (E_a - \mu Q_a)] \}^q, \quad (37)$$

we have

$$P_N = \frac{Z_q^{(N)}}{Z_q}. \quad (38)$$

Using the integral representation of the q -exponential function, we can write

$$Z_q^{(N)}(\beta, \mu, V) = \int_0^\infty dx G(x) Z^{(N)}(x\beta, \mu, V) \quad (39)$$

where $Z^{(N)}$ is the Boltzmann-Gibbs restricted partition function

$$Z^{(N)}(\beta, \mu, V) = \sum_a \delta(N - N_a) \exp [-\beta (E_a - \mu Q_a)]. \quad (40)$$

It is useful to introduce the generating function for the multiplicity distribution, defined by

$$F(t) \equiv \sum_{N=0}^{\infty} t^N P_N. \quad (41)$$

For example, if P_N is a Poisson distribution,

$$P_N = \frac{\bar{N}^N}{N!} e^{-\bar{N}}, \quad (42)$$

then its generating function is an exponential,

$$F_P(t) = \exp[\bar{N}(t-1)]. \quad (43)$$

For the negative binomial distribution, Eq. (1), we get

$$F_{NB}(t) = \left[1 - \frac{\bar{N}}{k} (t-1) \right]^{-k} = \exp_q [\bar{N}(t-1)], \quad (44)$$

where $q = 1 + 1/k$. It is suggestive to note that the negative binomial generating function is obtained substituting the exponential in the Poisson generating function by the q -exponential. For large values of k the negative binomial generating function has the asymptotic behavior

$$F_{NB}(t) \simeq \exp \left[\bar{N}(t-1) + \frac{\bar{N}^2}{2k} (t-1)^2 + \dots \right], \quad (45)$$

a result that will be useful later on. We see that in the limit $k \rightarrow \infty$ the negative binomial reduces to the Poisson distribution.

5 Relativistic ideal gas

In order to study the Tsallis statistics of a hadronic gas, it is tempting to apply the results of the previous sections to an ideal relativistic Boltzmann gas. Let us consider h different particle species, with masses m_i and charges q_i , $i = 1 \dots h$. In this case, the Boltzmann-Gibbs partition function is

$$Z(\beta, \mu, V) = \exp \left(V \sum_{i=1}^h \Phi_i(\beta) \exp(\beta \mu q_i) \right), \quad (46)$$

where

$$\Phi_i(\beta) = \frac{g_i}{2\pi^2} \frac{m_i^2}{\beta} K_2(\beta m_i). \quad (47)$$

Above, g_i is the statistical factor of particle i and $K_2(z)$ is a modified Bessel function.

The N -particle partition function is

$$Z^{(N)}(\beta, \mu, V) = \frac{1}{N!} \left(V \sum_{i=1}^h \Phi_i(\beta) \exp(\beta \mu q_i) \right)^N \quad (48)$$

and, from Eq. (38), the multiplicity distribution of the ideal Boltzmann gas is seen to be Poissonian. The average number of particles is

$$\overline{N} = n(\beta, \mu) V, \quad (49)$$

where

$$n(\beta, \mu) = \sum_{i=1}^h \Phi_i(\beta) \exp(\beta \mu q_i) \quad (50)$$

is the particle density.

We meet a problem when we try to apply Tsallis' statistics to the ideal gas: for $q > 1$ the generalized partition function Z_q is infinite. The integral representation of Z_q , Eq. (28), diverges if the ideal gas partition function (46) is substituted in the integrand. This happens because for $\beta \rightarrow 0$,

$$\Phi_i(\beta) \simeq \frac{g_i}{\pi^2 \beta^3}, \quad (51)$$

so that $Z(x\beta, \mu, V)$ has an essential singularity at $x = 0$ which cannot be removed by any power of x in $G(x)$. Therefore, the q -statistics of an ideal relativistic gas cannot be defined — there is no ideal Tsallis gas with $q > 1$.

6 Relativistic van der Waals gas

When restricted to N particles, the generalized ideal gas partition function $Z_q^{(N)}$ has a well defined integral representation in terms of the corresponding

Boltzmann-Gibbs function (Eq. (48)), provided

$$N < \frac{1}{3(q-1)} . \quad (52)$$

It is the production of particles above this limit that causes the divergence in the partition function of the Tsallis ideal gas. High multiplicities can be suppressed if a repulsive interaction exists among the produced particles. A simple way to include such interactions is to introduce a “van der Waals” excluded volume simulating the effect of hard-core potentials among particles. The van der Waals gas model has been frequently used to analyze particle abundances in high-energy heavy ion collisions [22, 7].

Let v_0 be the excluded volume associated to a particle. The corresponding partition function for N particles is obtained by replacing the volume of the system V by $V - Nv_0$,

$$Z^{(N)}(\beta, \mu, V) \rightarrow Z^{(N)}(\beta, \mu, V - Nv_0) \Theta(V - Nv_0) . \quad (53)$$

The Heaviside Θ -function limits the number of particles inside the volume V to $N < V/v_0$. The partition function for the relativistic van der Waals gas is then [22]

$$Z(\beta, \mu, V) = \sum_N \frac{1}{N!} n(\beta, \mu)^N (V - Nv_0)^N \Theta(V - Nv_0) \quad (54)$$

where $n(\beta, \mu)$ is the ideal gas density given in Eq. (50). In the large V asymptotic limit, this can be written as

$$Z(\beta, \mu, V) = \exp \left\{ \frac{V}{v_0} W[v_0 n(\beta, \mu)] \right\} , \quad (55)$$

where $W(x)$ is the Lambert function [23, 24], defined by the equation

$$W(x)e^{W(x)} = x . \quad (56)$$

Similarly, the generating function of the excluded volume gas is found to be

$$F(t) = \frac{1}{Z(\beta, \mu, V)} \exp \left\{ \frac{V}{v_0} W[t v_0 n(\beta, \mu)] \right\} . \quad (57)$$

The Lambert function $W(z)$ has the asymptotic behavior, in the limit $z \rightarrow \infty$,

$$W(z) \simeq \ln z + \ln \ln z + \dots . \quad (58)$$

The essential singularity at $\beta \rightarrow 0$ of the ideal Boltzmann gas partition function is thus reduced to a power one,

$$Z(\beta, \mu, V) \sim \beta^{-3V/v_0} . \quad (59)$$

Therefore, the integral in

$$Z_q(\beta, \mu, V) = \int_0^\infty dx G(x) \exp \left\{ \frac{V}{v_0} W[v_0 n(x\beta, \mu)] \right\} , \quad (60)$$

converges as long as

$$q < 1 + \frac{v_0}{3V}, \quad (61)$$

as expected from Eq. (52), since the hard core suppresses the production of particles above the maximum number V/v_0 . In this range of q values, the generating function of the multiplicity distribution in the Tsallis - van der Waals gas is given by

$$F(t) = \frac{1}{Z_q(\beta, \mu, V)} \int_0^\infty dx G(x) \exp \left\{ \frac{V}{v_0} W[t v_0 n(x\beta, \mu)] \right\}. \quad (62)$$

7 Tsallis and van der Waals corrections to the ideal gas

In order to examine in more detail the effect of Tsallis statistics on the multiplicity distribution, let us consider the case in which $q - 1$ and v_0 are both small (respecting the limit of Eq. (61)). The Lambert function $W(z)$ has the series expansion [23]

$$W(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n, \quad (63)$$

so that for $v_0 \rightarrow 0$ we have

$$W(tv_0 n) \simeq tv_0 n - (tv_0 n)^2 + \dots \quad (64)$$

and the generating function for the van der Waals-Tsallis relativistic gas can be written as

$$F(t) \simeq \frac{1}{Z_q} \int_0^\infty dx G(x) \exp \{ t V n(x\beta, \mu) [1 - tv_0 n(x\beta, \mu)] \}. \quad (65)$$

For $q \rightarrow 1$, this integral can be calculated with the help of Eq. (32). To first order in $q - 1$ and v_0 we obtain

$$F(t) \simeq \exp \{ (t-1) V n [1 + (q-1) \lambda (V n \lambda - 1) - 2v_0 n] + (t-1)^2 (V n)^2 [(q-1) \lambda^2 / 2 - v_0 / V] \} \quad (66)$$

where

$$\lambda(\beta, \mu) = -\frac{\beta}{n} \frac{\partial n}{\partial \beta}. \quad (67)$$

Comparison to Eq. (45) shows that this is the generating function of a negative binomial distribution, with average number of particles

$$\overline{N} = V n [1 + (q-1) \lambda (V n \lambda - 1) - 2v_0 n] \quad (68)$$

and a (large) k -parameter given by

$$\frac{1}{k} = (q-1) \lambda^2 - 2 \frac{v_0}{V}. \quad (69)$$

Strictly speaking, a negative binomial distribution is obtained only if $k > 0$, or

$$q > 1 + \frac{2v_0}{\lambda^2 V} . \quad (70)$$

In most cases of interest for hadron production the value of $\lambda(\beta, \mu)$ is large, so that Eqs. (70) and (61) can both hold. If q is below the limit of Eq. (70), k is negative, and the multiplicity distribution will have a binomial-like form. From Eq. (69) we see that the effect of Tsallis statistics is to make the multiplicity distribution wider than the Poisson distribution with the same average number of particles. The effect of the van der Waals excluded volume is to reduce the width relative to Poisson.

8 Charged particle multiplicity in hadronic collisions

Pions are the most copiously produced particles in high-energy hadronic collisions. They come not only from the thermally equilibrated gas, but also from the decay of heavier hadrons formed in the gas. A precise description of particle multiplicities has to take into account how much these unstable hadrons, called resonances, participate in the production of the observed particles. In the present analysis, we intend to understand the basic influence of the q -statistics on the particle multiplicity distribution. Therefore, to simplify the picture, we will study the behavior of the lowest-lying non-strange mesons, π , η , ρ and ω in thermal equilibrium. The η , ρ and ω mesons are resonances which decay into pions. The essential features of the baryon-free hadronic gas can be described by the mixture of these mesons, and the major conclusions of the present work will not be changed by this simplification.

In Fig. 1, we show the multiplicity distribution of charged particles produced in pp collisions at momentum $p_{lab} = 250$ GeV/c. Triangles are data from the NA22 experiment [25]. The average number of charged particles is $\overline{N} = 7.88 \pm 0.09$ and the width of the distribution is $D = 4.10 \pm 0.05$. The lines are multiplicity distributions calculated for the meson system described above, treated as an ideal Boltzmann gas. Two different temperatures were considered — $T = 190$ MeV (solid line), and $T = 160$ MeV (dashed line) — corresponding to the range of values found in analyses of pp reactions [26]. In both cases the volume of the gas was chosen such as to reproduce the measured average multiplicity. For the higher temperature this gives $V = 18.8$ fm³, and for the lower, $V = 43.0$ fm³. We assumed that the meson gas has a total electric charge $Q = 1$, which is the typical value in pp collisions. Changing this charge to 0 or 2 has only a small effect on the multiplicity distribution. All calculations were performed in the canonical framework.

Due to resonance decay, and exact charge conservation in the canonical ensemble, the calculated multiplicity does not follow exactly the Poisson distribution obtained in the grand canonical approach of Sec. 5. The deviation from

Poisson is conveniently measured by the quantity

$$\frac{1}{k} = \frac{D^2 - \overline{N}}{\overline{N}^2}, \quad (71)$$

a definition motivated by the k -parameter of the negative binomial distribution (compare with Eq. (2)). The ideal gas distributions shown in Fig. 1 have $k \sim 40$ and are, therefore, wider than Poisson. They are well approximated by a negative binomial distribution with the same k parameter. The large value of k shows that the deviation from Poisson caused by resonance decay and charge conservation is relatively small. This is reflected in Fig. 1, where multiplicity distributions constrained to have the same average present similar widths, despite the very different gas temperatures and volumes. It also indicates that the ideal gas model cannot explain the multiplicity data, in particular its large width. For all reasonable values of T and V consistent with the experimental average multiplicity, the ideal gas shows approximately the same narrow distribution. One may think that including more hadronic resonances in the model improves the situation. However, Becattini *et al.* [5] find $k > 10$ in their analysis of e^+e^- data, which includes many resonances. Hadronic collisions produce somewhat wider multiplicity distributions; the NA22 data shown in Fig. 1 have $k = 7$, and as the energy increases k becomes smaller [27]. We thus conclude that the multiplicity distribution of the ideal relativistic gas is not consistent with the hadron-hadron experimental data.

The situation doesn't get better if the meson system is treated as a relativistic van der Waals gas. This is seen in Fig. 2, where the NA22 data is compared to the canonical multiplicity distribution of the meson gas with excluded volume $v_0 = 0.368 \text{ fm}^3$. Again two temperatures, $T = 190 \text{ MeV}$ (solid line) and 160 MeV (dashed line), were considered, with volumes $V = 24.5 \text{ fm}^3$ and 49.2 fm^3 , respectively, determined by the experimental average multiplicity. The excluded volume reduces the width of the multiplicity distribution, relative to the ideal gas with the same average multiplicity; we have $k \sim 100$ for the van der Waals gas distributions shown in Fig. 2. The width reduction occurs for any value of v_0 . Therefore, the van der Waals gas model cannot explain the experimental multiplicity distribution; it is even less satisfactory than the ideal gas model.

Tsallis statistics for $q > 1$ produces multiplicity distributions that are wider than the Boltzmann-Gibbs ones (see Sec. 7). In Fig. 3, we show some examples of the canonical multiplicity distribution obtained in q -statistics. The solid line corresponds to a gas with $\beta^{-1} = 190 \text{ MeV}$, $V = 9.13 \text{ fm}^3$, $q = 1.0097$, and the dashed line to $\beta^{-1} = 160 \text{ MeV}$, $V = 31.0 \text{ fm}^3$, $q = 1.0030$. In both calculations the excluded volume is $v_0 = 0.368 \text{ fm}^3$. We note that a even a small deviation of q from unity causes a substantial broadening of the distribution, taking it much closer to the experimental data. Thus, it seems possible to give a thermostatistical description of high-energy hadronic multiplicities, provided nonextensivity effects are taken into account.

9 Concluding remarks

In the present work we have studied the effect of the Tsallis nonextensive statistics (with $q > 1$) on the multiplicity distribution of particles in a high-energy gas. In order to avoid divergences in the generalized partition function of the ideal gas, a van der Waals hard-core interaction was introduced as a physical regularization procedure. Nonextensive statistics gives rise to multiplicity distributions that are broader than the Boltzmann-Gibbs ones. In the small $q - 1$ limit, the Tsallis multiplicity distribution can be approximated by a negative binomial distribution, with k -parameter proportional to $(q - 1)^{-1}$. We developed a simple meson gas model in order to compare the different statistics to experimental data from hadronic collisions. The multiplicity distribution obtained with the Boltzmann-Gibbs statistics has a too narrow width compared to the experimental results. A small increase of q from unity makes the distribution considerably wider, and a good agreement with the data can be obtained within Tsallis statistics. These results provide an interesting perspective on why hadronic multiplicities have a negative binomial distribution, and suggest that nonextensive thermal phenomena play an important role in high-energy collisions.

In order to get a more complete picture of the nonextensive thermostistical effects in hadron production, we need to extend our analysis using a larger resonance gas, and including other observables such as particle ratios, transverse momentum spectra, and correlations. Work along these lines is in progress.

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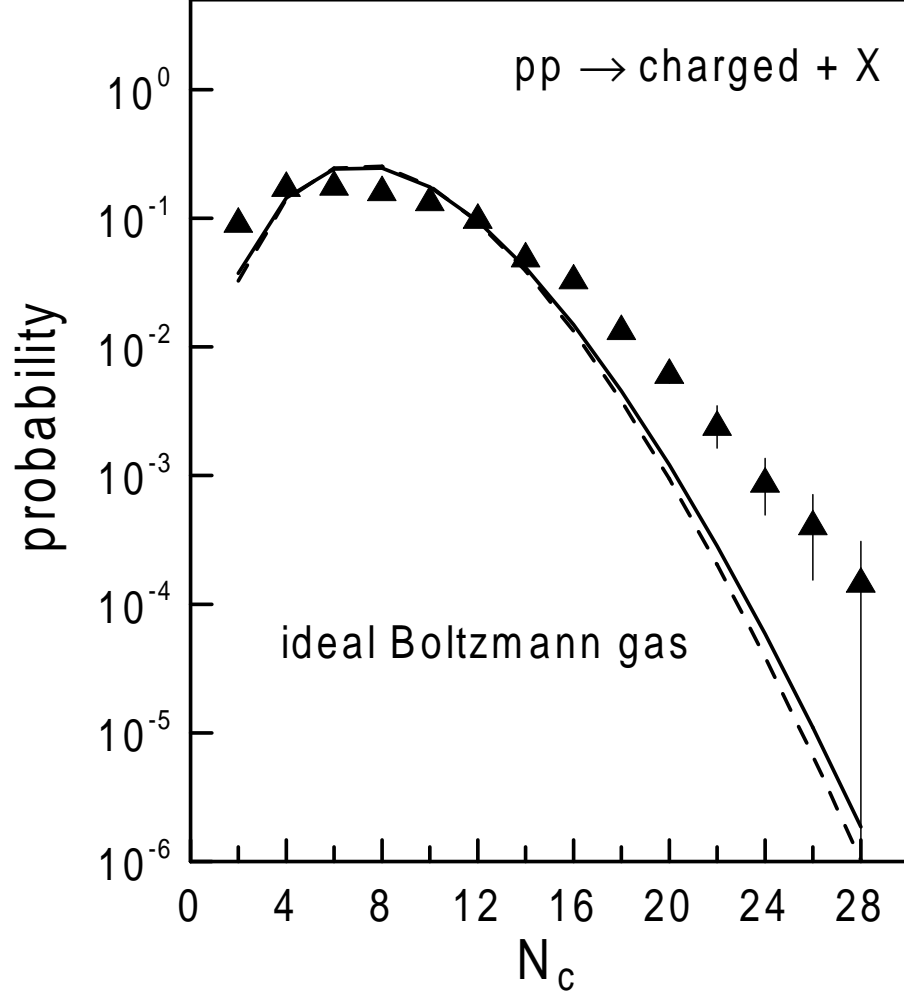


Figure 1: Multiplicity distribution of charged particles produced in pp collisions at 250 GeV/c. The lines are calculations based on the ideal Boltzmann gas model. The solid line corresponds to $T = 190$ MeV and $V = 18.84$ fm³, the dashed line to $T = 160$ MeV and $V = 42.96$ fm³

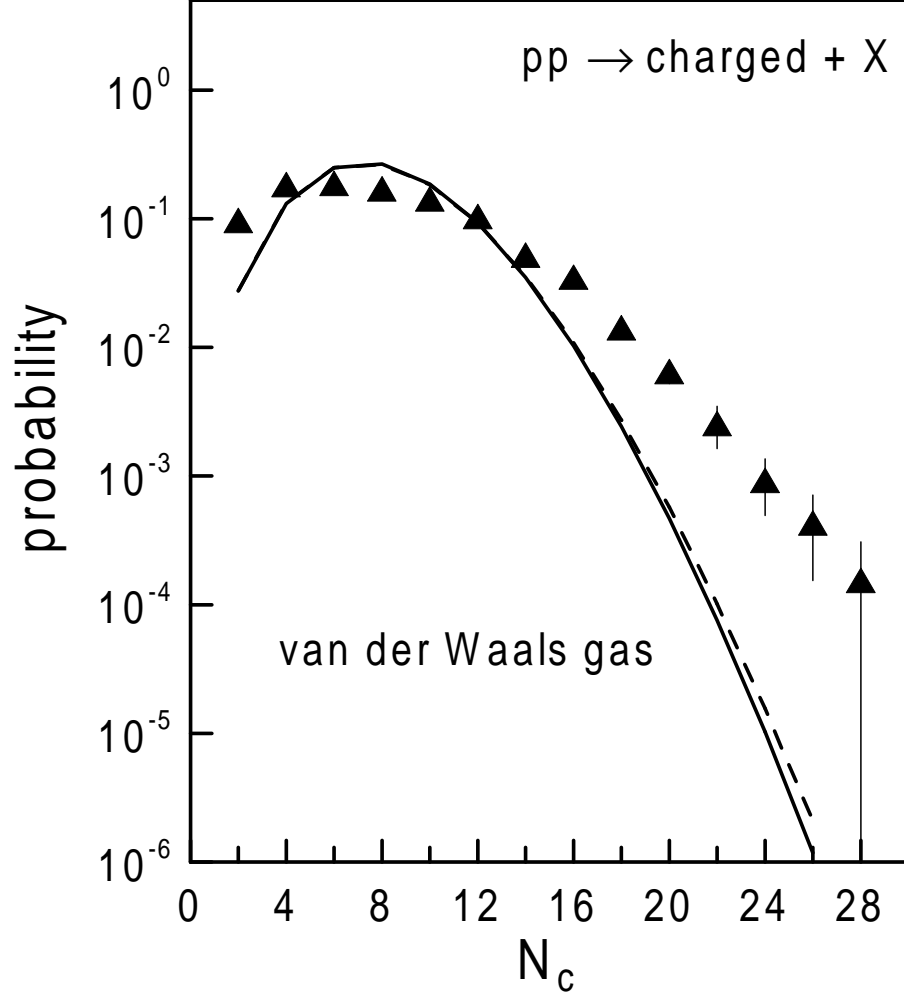


Figure 2: Charged particle multiplicity distribution in pp collisions at 250 GeV/c. The lines are multiplicity distribution of van der Waals gases with $T = 190$ MeV, $V = 24.46 \text{ fm}^3$ (solid), and $T = 160$ MeV, $V = 49.21 \text{ fm}^3$ (dashed). In both calculations the excluded volume is $v_0 = 0.368 \text{ fm}^3$.

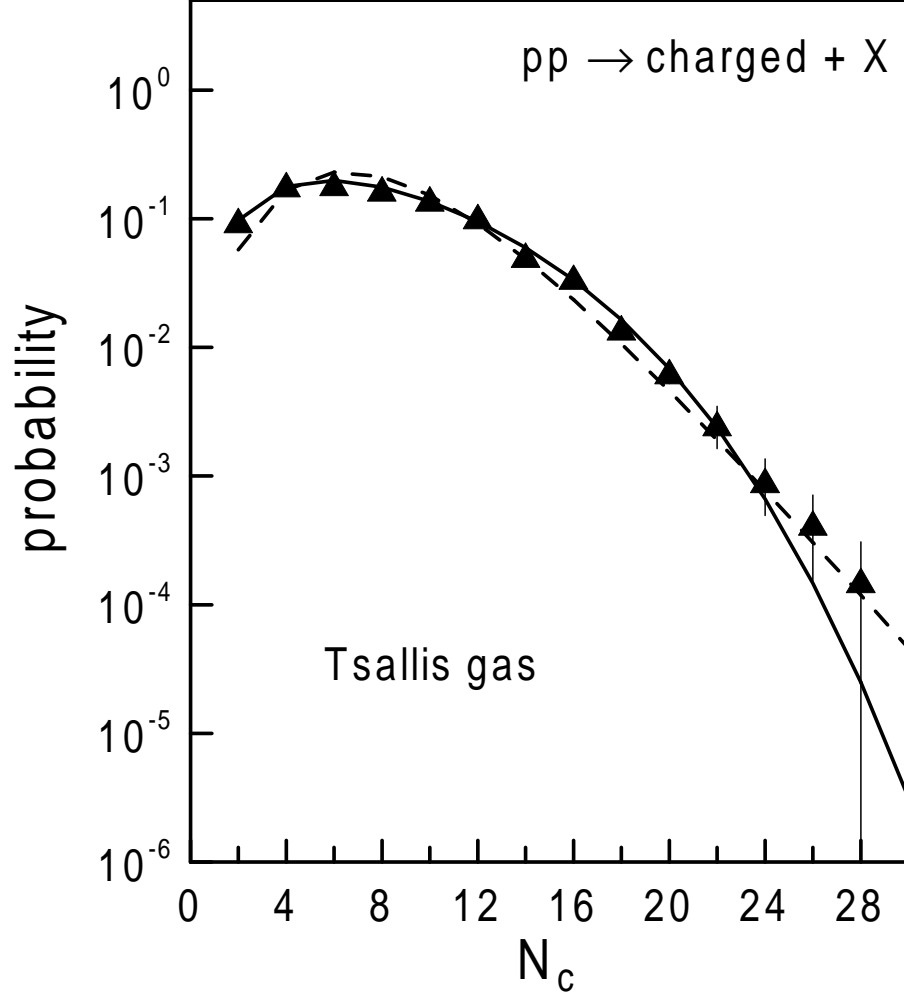


Figure 3: Charged particle multiplicity distribution in pp collisions at 250 GeV/c. The lines are multiplicity distribution of Tsallis gases with $\beta^{-1} = 190$ MeV, $V = 9.13$ fm³, $q = 1.0097$ (solid), and $\beta^{-1} = 160$ MeV, $V = 31.0$ fm³, $q = 1.0030$ (dashed). In both cases $v_0 = 0.368$ fm³.